PHYS 705: Classical Mechanics

HW#8 (11/8) and #9 (11/15) move down one more week

HW#8 and #9: CT and Hamilton-Jacobi Eq

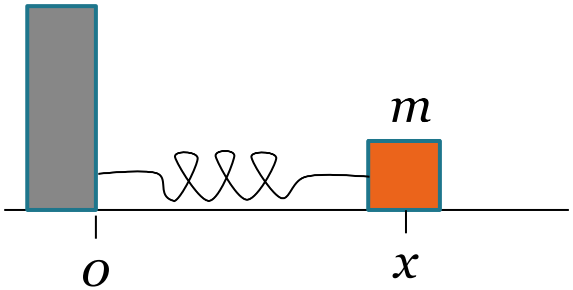
HW#10: Small Oscillations

HW#11: Noninertial Reference Frame and Rigid Body Motion

HW#12: Rigid Body Motion (more practice problems)

Review from Previous Lecture

Example: Harmonic Oscillator



$$f(x) = -kx \qquad U(x) = \frac{kx^2}{2}$$

$$L = T - U = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$$

Since U does not dep on $\dot{x}$ and $x \mapsto x$ does not dep t explicitly, $H = E$.

Define $\omega = \sqrt{k/m}$ or $\omega^2 m = k$ → $$\boxed{H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)}$$

Notice that the system is not cyclic with the original variable x. We will attempt to find a canonical transformation from (x, p) to (X, P) such that X is cyclic so that P is constant.

Example: Harmonic Oscillator

With the form of H as sums of squares $H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)$, we will try to exploit $\sin^2 + \cos^2 = 1$

Trail Solution: $p = g\left(P\right)\cos\left(X\right)$ $\quad x = \frac{g\left(P\right)}{m\omega}\sin\left(X\right)$ (*)

Substituting these into our Hamiltonian, we have

X is cyclic in K !

$$K = \frac{1}{2m}\left(g^2\left(P\right)\cos^2 X + m^2\omega^2 \frac{g^2\left(P\right)}{m^2\omega^2}\sin^2 X\right) = \frac{g^2\left(P\right)}{2m}$$

Example: Harmonic Oscillator

Need to find the right $g(P)$ such that the transformation is canonical!

Try the following type 1 generating function: $F = F(x, X, t)$

(with x and X being the indep vars)

For F to be canonical transformation, $F(x, X, t)$ must satisfy the partial derivative relations (Type 1 in Table 9.1 in G):

$$p = \frac{\partial F}{\partial x} \qquad\qquad P = -\frac{\partial F}{\partial X}$$

Example: Harmonic Oscillator

Dividing the two trial solutions, we get,

$$\frac{p}{x} = \frac{\cancel{g(P)}\cos X}{\cancel{g(P)}\sin X\,(1/m\omega)} = m\omega\cot X \qquad \text{or} \qquad p = m\omega x\cot X$$

Then, the two partial derivative equations give:

$$p = \frac{\partial F}{\partial x} = m\omega x\cot X \quad \Rightarrow \quad F = F(x, X, t) = \frac{m\omega x^2}{2}\cot X$$

$$P = -\frac{\partial F}{\partial X} \quad \Rightarrow \quad P = \frac{m\omega x^2}{2}\frac{1}{\sin^2 X}$$

⋮

Example: Harmonic Oscillator

$$\vdots$$

$$x = \sqrt{\frac{2P}{m\omega}} \sin X \qquad p = \sqrt{2m\omega P} \cos X$$

By comparing with our trial transformation equation,

$$\begin{cases} x = \dfrac{g(P)}{m\omega} \sin X \\ p = g(P) \cos X \end{cases} \quad \text{gives} \quad g(P) = \sqrt{2m\omega P} \quad (*')$$

Then, the transformed Hamiltonian is: $K = \dfrac{g^2(P)}{2m} = \dfrac{2m\omega P}{2m} = \omega P$

Example: Harmonic Oscillator

$\frac{\partial F}{\partial t} = 0$

With the transformed Hamiltonian $K = \omega P = H = E$, the Hamilton's Equations give,

$$\dot{P} = -\frac{\partial K}{\partial X} = 0 \quad (X \text{ is cyclic}) \qquad \dot{X} = \frac{\partial K}{\partial P} = \omega$$

$$\rightarrow \boxed{P = const} \qquad \rightarrow \boxed{X = \omega t + \alpha}$$

depends on IC

Together with the inverse transform, $x(t), p(t)$ can be solved explicitly,

$$x = \sqrt{\frac{2P}{m\omega}} \sin(\omega t + \alpha)$$

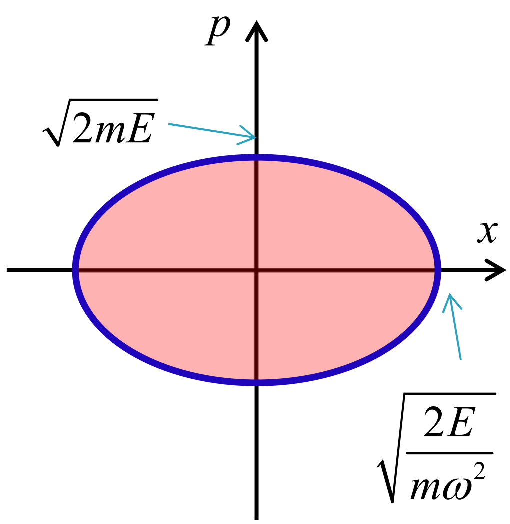
$$p = \sqrt{2m\omega P} \cos(\omega t + \alpha)$$

Example: Harmonic Oscillator

Then with $K = \omega P = E$

$$x = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)$$

$$p = \sqrt{2mE} \cos(\omega t + \alpha)$$

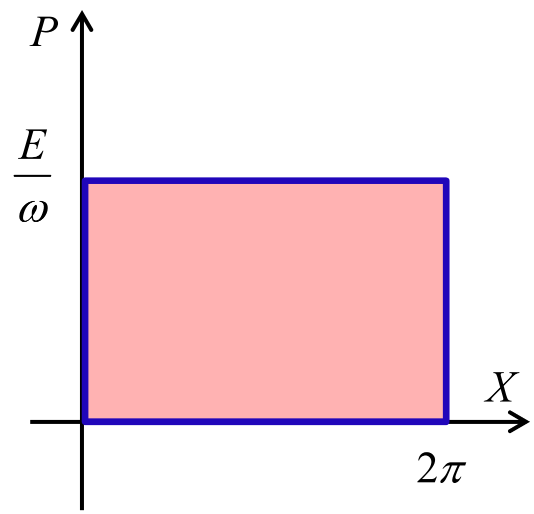


Phase Space area is invariant!

$$area = \frac{2\pi E}{\omega}$$

(X, P) is an example of Action-Angle variable pair,

$$X = \omega t + \alpha \qquad P = const$$



"Symplectic" Approach & Poisson Bracket

By considering $Q_j(q,p)$ and $P_j(q,p)$ as function of q and p explicitly and they satisfy the Hamilton's Equations,

We can derive the following four "direct conditions" for various (i,j) for the canonical transformation pair, $Q_j(q,p), P_j(q,p)$.

$$\left(\frac{\partial Q_j}{\partial q_i}\right)_{q,p} = \left(\frac{\partial p_i}{\partial P_j}\right)_{Q,P} \qquad \left(\frac{\partial Q_j}{\partial p_i}\right)_{q,p} = -\left(\frac{\partial q_i}{\partial P_j}\right)_{Q,P}$$

$$\left(\frac{\partial P_j}{\partial q_i}\right)_{q,p} = -\left(\frac{\partial p_i}{\partial Q_j}\right)_{Q,P} \qquad \left(\frac{\partial P_j}{\partial p_i}\right)_{q,p} = \left(\frac{\partial q_i}{\partial Q_j}\right)_{Q,P}$$